

Dimensional Reduction and Quantum-to-Classical Reduction at High Temperatures

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Abstract

We discuss the relation between dimensional reduction in quantum field theories at finite temperature and a familiar quantum mechanical phenomenon that quantum effects become negligible at high temperatures. Fermi and Bose fields are compared in this respect. We show that decoupling of fermions from the dimensionally reduced theory can be related to the non-existence of classical statistics for a Fermi field.

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I. INTRODUCTION

The phenomenon of dimensional reduction in finite temperature field theories has attracted considerable interest recently. This phenomenon simplifies the description of finite temperature phase transitions such as, for example, the chiral symmetry restoration transition in QCD [1], or the deconfinement transition in pure gauge theories [2]. It helps to understand the behavior of QCD at $T \gg T_c$ as well [3]. The study of the electroweak phase transition has been also focused on the issue of dimensional reduction recently [4].

Let us first briefly present the common view on the dimensional reduction existing in the literature. The traditional description is based on the Euclidean formalism. In this approach a given system is living in a box of $d + 1$ dimensions. The extent in d spatial dimensions is much larger than the physical scale (correlation length in the system). The extent in the $d + 1$ -th (Euclidean time) dimension is $1/T$, where T is the temperature. The boundary conditions in this time direction are periodic for Bose fields and antiperiodic for Fermi fields.

Perturbation theory at finite temperature¹ contains integrals over spatial d -momenta and (infinite) sums over discrete Matsubara frequencies ω_n . Such a sum can be interpreted as a sum over particles with masses ω_n in the perturbation theory for some d -dimensional field theory at zero temperature. One can argue then that the particles with high Matsubara masses can be integrated out and low momentum physics can be described by a d -dimensional effective theory for particles with smallest Matsubara masses. The effect of other Matsubara modes is to renormalize the couplings of this effective theory. For a Bose field the Matsubara masses are quantized as even multiples of πT . Thus, one can describe physics at momentum scales much less than T by an effective d -dimensional theory of the $\omega = 0$ Matsubara mode. Matsubara masses for fermions are odd multiples of πT and thus at the scale much less than

¹ The argument of this paragraph applies to theories interacting sufficiently weakly. Also, masses of the particles are neglected compared to T .

T they decouple.

The dimensional reduction has a simple geometric origin. Fluctuations with wavelength larger than the extent in the Euclidean time direction $1/T$ are “squeezed” in this direction and behave like d -dimensional fluctuations. In other words, for such fluctuations the system looks like a “pancake”. If such fluctuations determine the physics of a phase transition (critical behavior) then this physics is effectively d -dimensional.

In this paper we wish to emphasize the fact that these long wavelength fluctuations are simply classical thermal fluctuations. Such a view provides a nice physical interpretation of the dimensional reduction as a well known phenomenon that quantum effects can be neglected at high temperatures. In essence, one uses the fact that quantum oscillators with frequencies ω behave as classical when² $\hbar\omega \ll k_B T$, i.e., when thermal energies are much larger than the typical distance between quantum levels.³ In quantum field theory the corresponding situation can occur either as $T \rightarrow \infty$ or at a second order phase transition as $T \rightarrow T_c$ and $\omega \rightarrow 0$.

Such an interpretation of the dimensional reduction is missing in the current literature on this subject [1–4,6]. We do not attempt to rederive results of the traditional approach to dimensional reduction. The motivation for this note is to show a simple and rather familiar physical picture behind the phenomenon which is largely viewed as a technical trick. In the context of condensed matter phenomena such a physical picture is rather straightforward and quantum versus classical transitions have been a subject of research [7]. However, in

² Later on we shall use units in which $k_B = 1$ and $\hbar = 1$.

³ In quantum mechanics of one degree of freedom x the reduction to classical statistics at high temperatures can be viewed formally as a dimensional reduction at $d = 0$. Explicit relation between quantum and classical partition functions at high T is discussed in the path integral formalism in [5]. The functional integral over trajectories $x(t)$ (imaginary t) in quantum partition function reduces to an integral over x .

such a context the relation to the issue of dimensional reduction is obscured by the absence of relativistic (space–time) invariance.

In Section II we discuss how in the case of a free or weakly interacting Bose field at finite temperature the classical statistical behavior of low momentum modes is related to the dimensional reduction. Nothing is new there. We only wish to point out the relation between the dimensional reduction and this well-known statistical behavior of bosonic fields. Similar discussion is given for weakly interacting Fermi fields in Section III. We relate the non-existence of classical statistics for Fermi fields to the decoupling of fermions in the dimensionally reduced theory.

We discuss weakly coupled theories partly because the traditional Euclidean Matsubara masses approach to dimensional reduction with which we wish to make contact applies only then. More general discussion is given at the end of Section IV.

II. BOSE FIELDS

Consider a quantum theory of some Bose field. For simplicity consider a massless field. Such a limit is useful for studying critical phenomena in scalar theories near the finite T phase transition, when the (T dependent) mass is small compared to T . We shall consider a noninteracting or weakly interacting theory. In fact, for our purposes a photon gas is a good example.

Our field theory can be viewed as a set of noninteracting (or very weakly interacting) harmonic oscillators – momentum modes – with frequencies ω . At finite T each oscillator has average energy according to Planck:

$$\epsilon(\omega) = \frac{\omega}{\exp(\omega/T) - 1}. \quad (2.1)$$

This corresponds to the average occupancy of each mode by the photons (or whatever bosons):

$$n(\omega) = \frac{1}{\exp(\omega/T) - 1}. \quad (2.2)$$

Now we note that oscillators with frequencies $\omega \ll T$ are excited to very high energy levels $n \gg 1$. For such oscillators, as we know, quantum effects are small. Thus the modes with $\omega \ll T$ can be described by a classical (statistical) theory. For example, from equation (2.1) we can find that in such a limit $\epsilon \approx T$ in accordance with a general theorem of classical statistical mechanics (equipartition of energy).⁴ For the photon gas this is the Rayleigh–Jeans region of frequencies. The classical statistics/thermodynamics of a field represented by these modes is what the dimensionally reduced theory is.

Where are the Matsubara frequencies in this picture? They are given by the poles in the distribution (2.2). They are imaginary except $\omega = 0$. This pole is responsible for the equipartition of energy at small ω : $\epsilon = \omega n(\omega) \approx T$.

The classical thermodynamics of the photon gas is inconsistent because of the ultraviolet catastrophe: the total energy (or, specific heat) diverges because of the contribution of high frequency modes. This means that such a theory can be only an *effective theory* for the modes with frequencies smaller than $O(T)$. The UV divergence is regularized by quantum effects.

III. FERMI FIELDS

Next, consider a (free or weakly interacting) theory of fermionic excitations. For reasons similar to the bosonic case we neglect the mass of the fermions. For a Fermi field the momentum modes cannot be viewed as quantum oscillators, rather they are two-level systems. The average energy and the occupation number of each mode are given by:

$$\epsilon_f(\omega) = \frac{\omega}{\exp(\omega/T) + 1}, \quad (3.1)$$

$$n_f(\omega) = \frac{1}{\exp(\omega/T) + 1}. \quad (3.2)$$

⁴A classical oscillator has two degrees of freedom: the coordinate and the momentum.

The modes with small ω do not behave classically at all. If such a mode could be represented by a classical system such a system would have less than one degree of freedom: $\epsilon \approx \omega/2$ for $\omega \ll T$. Of course, this is due to the fact that a fermion level cannot be occupied by more than one fermion: $n_f \leq 1$, and coherent classical Fermi fields do not exist. Again, the Matsubara frequencies are the poles of $n_f(\omega)$. The absence of a pole at $\omega = 0$ is the reason for the non-classical behavior.

The critical behavior at a second order phase transition is determined by long wavelength (low ω) fluctuations. From (2.1) and (3.1) we see that the contribution of a Fermi field to the total energy is much smaller than the contribution of a Bose one at very small ω .⁵ This is related to the decoupling of fermions in the formalism based on Matsubara masses.

What is the meaning of masslessness of these fermions then? One can see explicitly that there is no pole at $p = 0$ in the (free) fermion propagator at finite T for *massless* fermions. One can also calculate the free propagator in the coordinate space and see that it decays exponentially with spatial separation r as $\exp(-Tr/\pi)$. What happens to the pole at $p = 0$ which exists at zero T ?

It is instructive to see this in the following way. Consider a (hermitian) perturbation V creating/annihilating a fermion with momentum \mathbf{p} . The propagator is proportional to the response of the system (e.g., change of the free energy) to this perturbation. The mode with momentum \mathbf{p} is a two-level system with energies: $E_0 = 0$ and $E_1 = |\mathbf{p}|$. The levels are “repulsed” under the perturbation V . To the order V^2 the shifts are given by:

⁵ More precisely, a low frequency ($\omega_p \ll T$) bosonic mode contributes to the free energy: $T \ln(\omega_p/T)$, while a fermionic one contributes: $\omega_p/2 - T \ln 2$. The latter expression is the energy minus the entropy for a two-level system. For the bosonic mode the entropy dominates in the free energy. If we take $\omega_p^2 = a(T - T_c) + p^2$ (Gaussian model) near a phase transition at $T = T_c$, we find that the contribution to the specific heat $C_V = -T(\partial^2 F/\partial T^2)$ is proportional to ω_p^{-4} for bosons and to ω_p^{-3} for fermions.

$$\Delta E_0 = \frac{\langle 0|V|1\rangle\langle 1|V|0\rangle}{E_0 - E_1}, \quad \Delta E_1 = \frac{\langle 1|V|0\rangle\langle 0|V|1\rangle}{E_1 - E_0} \quad (3.3)$$

The shift of E_0 is due to a virtual process in which the particle is created from the vacuum and then annihilated. The shift of E_1 is due to a process in which the particle is annihilated and then created back. The amplitudes for these processes are equal because V is hermitian. Hence the level shifts have equal magnitudes and opposite signs.

Only the first process contributes to the propagator at zero temperature. The pole at $\mathbf{p} = 0$ is due to the one-massless-particle intermediate state $|1\rangle$. At nonzero temperature the second process (with vacuum intermediate state) also contributes to the response of the system because there are particles in the thermal bath already. Its contribution cancels the pole at $\mathbf{p} = 0$. Indeed, the change in the partition function Z under the perturbation V is given by⁶ :

$$-T\Delta Z = \Delta E_0 e^{-E_0/T} + \Delta E_1 e^{-E_1/T} = -\frac{|\langle 0|V|1\rangle|^2}{|\mathbf{p}|}(1 - e^{-|\mathbf{p}|/T}). \quad (3.4)$$

We see that although massless fermions are in the spectrum, they do not produce a pole at $\mathbf{p} = 0$ at finite T . Thus, for example, massless fermions do not induce long range exchange interactions at finite temperature.

IV. CONCLUSIONS AND DISCUSSION

The purpose of this note is to point out that dimensional reduction in quantum field theories can be understood as a familiar quantum mechanical phenomenon: quantum effects become negligible at high temperatures and classical statistics can be applied. The question of whether there is dimensional reduction at a given phase transition at finite T or there is

⁶ The eqs. (3.3), (3.4) constitute in essence the spectral representation of the propagator. Using this representation one can formally generalize the argument to theories where fermions interact not necessarily weakly.

not (as it is discussed, for example, in the case of the electroweak transition [4]) is equivalent to the following question: Is the physics of the transition classical or quantum? In other words, can the transition be described by some classical statistical field theory or not?

In pure bosonic theories one can expect the answer to be yes if the transition is of the second order. The physics of such a transition is determined by long wavelength (low frequency) fluctuations which are classical. The corresponding classical statistical field theory is what the dimensionally reduced theory will be.

In theories with fermions if the answer is to be yes, then the fermions must decouple: fermionic excitations are not classical.

If, for some reason, fermionic degrees of freedom are important at a phase transition then the transition is not classical. Such a possibility has been discussed recently for Yukawa and Gross-Neveu models [6]. However, this possibility seems unlikely at a *second* order phase transition at finite T . The physics of such a transition (critical behavior) is determined by low momentum excitations. Fermionic excitations of low momentum are suppressed by Fermi statistics (as discussed in Section III).

It might be possible to make a very general statement: Any second order phase transition at finite temperature is classical. From this point of view, the only possibility for transitions with non-negligible quantum effects (and thus possible importance of fermionic degrees of freedom) is provided by first order transitions (or, obviously, by transitions at exactly zero T).

Consider the hot electroweak phase transition as an example. The question of whether one can use dimensionally reduced theory to describe the transition or one cannot is, from a physical point of view, the question of whether the transition is classical or not. Qualitatively, the answer is simple: if the transition is of the second (or weakly first) order then the physics of the transition is dominated by classical thermal fluctuations. If the transition is strongly first order (which it is if the Higgs mass at zero T is small) then quantum effects are important at T_c and there is no dimensional reduction.

The discussion in this paper was mostly limited to theories which are free or interact-

ing weakly.⁷ In this case the traditional Euclidean Matsubara masses argument shows that a dimensionally reduced description exists for bosonic but not for fermionic low momentum modes. This corresponds, as we saw, to classical statistical behavior of bosonic low momentum modes and non-classical behavior of fermionic ones.

In a strongly coupled theory it could, in principle, depend on the given dynamics whether a dimensionally reduced Euclidean description exists at high temperature (or at a phase transition). By such a description we mean some *local* effective Euclidean theory in one less dimension. Simple arguments discussed here suggest that for theories interacting sufficiently weakly (at the energy scale of the order of T) such a description exists at a *second* order phase transition at finite temperature. It remains to be seen whether this conjecture holds for strongly coupled theories.

Nevertheless, it is clear that in *any* sensible case if a dimensionally reduced Euclidean theory for low momentum modes does exist, this reduced theory is simply a classical statistical theory. It is an *effective* theory and a cutoff of order T should be implemented (like, e.g., in the UV catastrophe). The effective classical degrees of freedom (collective excitations), the thermal masses and the couplings, can non-trivially depend on and should be derived from the underlying *quantum* (thermo)dynamics.

In weakly coupled theories perturbation theory can be used for such a derivation, although the resulting classical statistics (critical behavior) is non-perturbative. A typical example is $\lambda\phi^4$ theory with $\lambda \ll 1$. It is convenient to view the thermal mass $m(T)$ as a measure of the distance from the criticality: $m(T) \rightarrow 0$ as $T \rightarrow T_c$. Low momentum modes become classical when $m(T) \ll T_c$ while perturbation theory still works as long as $m(T) \gg \lambda T_c$. Similar situation occurs in the electroweak theory [4]. In strongly coupled theories, such as QCD at T_c , the effective classical theory is not derivable in such a pertur-

⁷ So that the representation of the field as a set of weakly coupled oscillators (or two-level systems) is possible.

bative way. Special methods and insights are necessary. For example, symmetry principles can be used to determine the relevant degrees of freedom of the effective theory [1,2].

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REFERENCES

- [1] R. Pisarski and F. Wilczek, Phys. Rev. **D29**, 338 (1984); F. Wilczek, Nucl. Phys. **A565**, 123c (1994).
- [2] B. Svetitsky and L.G. Yaffe, Nucl. Phys. **B210**, 423 (1982).
- [3] T. Appelquist and R. Pisarski, Phys. Rev. **D23**, 2305 (1981); S. Nadkarni, Phys. Rev. **D27**, 917 (1983); N.P. Landsman, Nucl. Phys. **B322**, 498 (1989); R.F. Alvarez-Estrada, Physica **A158**, 178 (1989); T. Reisz, Z. Phys. **C53**, 169 (1992).
- [4] For a review of electroweak phase transition at finite temperature and references see, e.g., K. Kajantie, Hot Electroweak Matter, a plenary talk at the ‘Lattice 94’ Conference, Bielefeld Sept. 27 – Oct. 1, 1994, to appear in Nucl. Phys. B (Proc. Suppl.), hep-lat/9412072.
- [5] R.P. Feynman, Statistical Mechanics (W.A. Benjamin, Inc., Reading, Massachusetts, 1972) Chapter 3.
- [6] B. Rosenstein, A.D. Spiliotopoulos and H.L. Yu, Phys. Rev. **D49**, 6822 (1994); A. Kocic and J. Kogut, Phys. Rev. Lett. **74**, 3109 (1995).
- [7] See, e.g., J.A. Hertz, Phys. Rev. **B14**, 1165 (1976); for a review see, e.g., A.P. Young, Quantum Phase Transitions, a plenary talk at the ‘Lattice 94’ Conference, Bielefeld Sept. 27 – Oct. 1, 1994, to appear in Nucl. Phys. B (Proc. Suppl.).